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LETTER TO THE EDITOR

Phase vortex spirals

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Abstract
For a complex scalar field $\psi(x, y)$ in the plane, the flow lines (integral curves of the current field Im $\psi^*\nabla\psi$) typically spiral slowly in or out of a phase vortex (where $\psi = 0$), with the distance between successive windings decreasing as $2\pi K r^3$ near the vortex at $r = 0$. The coefficient K depends on the derivatives of ψ at the vortex. In three dimensions, the flow spiral migrates slowly along the vortex line, in a helix whose pitch is proportional to r^2. For fields with well-defined orbital angular momentum, the flow lines can be determined explicitly not just near the vortex but also globally. The explicit forms of flow lines near phase extrema and saddles are also found.

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1. Introduction

The zeros of complex scalar functions ψ—representing waves, for example—are singularities where the phase of ψ is undefined [1–3]. Around each singularity, the current (orthogonal to the phase contours) circulates; therefore the zeros are phase vortices. It is known [4] that the flow lines approach circles close to a vortex.

My main purpose here is to study the details of this circulation. In the general case (section 2), an asymptotic averaging argument (section 2) shows that the flow lines spiral in or out of each vortex, with the spiralling getting more circular closer to the vortex. In an interesting class of special cases (section 3), the spiralling can be calculated without approximation. For completeness, the general form of the flow lines is also calculated (section 4) near stationary points of phase, that is phase extrema and phase saddles (at these stationary points, the phase contours are singular but the phase itself is well defined).

Consider functions in the plane

$$\mathbf{r} = \{x, y\} = r\{\cos\phi, \sin\phi\}, \quad (1)$$

represented either by their modulus and phase or their real and imaginary parts:

$$\psi(\mathbf{r}) = \rho(\mathbf{r}) \exp(\mathrm{i}\chi(\mathbf{r})) = u(\mathbf{r}) + \mathrm{i}v(\mathbf{r}). \quad (2)$$

Let the vortex be situated at the origin, i.e. $\psi(\mathbf{0}) = 0$. The waves could be confined to the $\mathbf{r}$ plane, or this plane could be a section through a wave in space—a situation we consider briefly at the end of section 2. Assume that $\psi(\mathbf{r})$ does not depend on time, except possibly through an overall phase factor (representing a monochromatic wave, for example), so the vortices do not move.

The current is the real vector field

$$\begin{aligned}\mathbf{j}(\mathbf{r}) &= \mathrm{Im}\,\psi^*(\mathbf{r})\nabla\psi(\mathbf{r}) = \rho(\mathbf{r})^2\nabla\chi(\mathbf{r}) \\ &= u(\mathbf{r})\nabla v(\mathbf{r}) - v(\mathbf{r})\nabla u(\mathbf{r}) = j_r(\mathbf{r})\mathbf{e}_r + j_\phi(\mathbf{r})\mathbf{e}_\phi.\end{aligned} \tag{3}$$

For a wave in space, $\mathbf{j}$ is the component of the three-dimensional current vector in the $\mathbf{r}$ plane. In quantum mechanics, $\mathbf{j}$ is the probability current, that is the expectation value of the local momentum operator. In an optical field well approximated by a scalar wave (for example a uniformly polarized paraxial wave), $\mathbf{j}$ is directed along the Poynting vector.

Our interest will be in the flow lines, defined as the integral curves of the vector field $\mathbf{j}$, satisfying the equation of motion

$$\dot{\mathbf{r}}(t) = \mathbf{j}(\mathbf{r}(t)). \tag{4}$$

Although not directly observable, the flow lines give a useful picture of the current distribution. (In the hydrodynamical interpretation of Madelung [5], the flow lines are the trajectories of the particles in an imaginary probability fluid; in the later and more popular Bohmian interpretation [6], the lines are regarded as trajectories of individual quantum particles.)

In the special case where $\nabla\cdot\mathbf{j} = 0$, the flow lines near a vortex are closed, because then $\mathbf{j}$ can be represented in the form $\nabla\times(f(\mathbf{r})\mathbf{e}_z)$, so the flow lines are the contour lines $f(\mathbf{r}) =$ constant. This occurs, for example, when $\nabla^2\psi(\mathbf{r}) = g(\mathbf{r})\psi(\mathbf{r})$, where $g(\mathbf{r})$ is a real function, as in the Schrödinger or Helmholtz equations in the plane. In the general case that we study here (for example waves in space, which do not satisfy a two-dimensional wave equation in the plane), the flow lines are not contour lines, and must be determined by integrating (4).

We wish to study the geometry of the flow lines, rather than the history of real or imaginary particles moving along them, so we eliminate time using (2) and (4), from which the radial and angular velocities are

$$\dot{r} = \frac{\mathbf{r}\cdot\mathbf{j}}{r}, \qquad \dot{\phi} = \frac{\mathbf{r}\times\mathbf{j}\cdot\mathbf{e}_z}{r^2}. \tag{5}$$

This leads to the equation for the geometry of the flow lines in polar coordinates:

$$\frac{\mathrm{d}r}{\mathrm{d}\phi} = \frac{r\,\mathbf{r}\cdot\mathbf{j}}{\mathbf{r}\times\mathbf{j}\cdot\mathbf{e}_z}. \tag{6}$$

2. Flow spirals near generic phase singularities

To determine the flow lines asymptotically close to the vortex at $\mathbf{r} = 0$, we expand ψ about its zero:

$$\begin{aligned}\psi(\mathbf{r}) &= c_i x_i + c_{ij}x_i x_j + c_{ijk}x_i x_j x_k + \cdots \\ &\equiv \mathbf{c}\cdot\mathbf{r} + \mathbf{r}\cdot\mathbf{C}\cdot\mathbf{r} + c_{ijk}x_i x_j x_k + \cdots.\end{aligned} \tag{7}$$

Here i, j, k take the values 1 (x component) or 2 (y component), and the summation convention is employed. $\mathbf{c}$ is a complex vector, $\mathbf{C}$ is a complex symmetric matrix and c_{ijk} is a complex fully symmetric third-rank tensor.

To leading order, in which $\mathbf{C}$ and c_{ijk} are neglected, $\mathbf{j} = \mathrm{Im}[\mathbf{c}^*\cdot\mathbf{r}\,\mathbf{c}]$, so the radial flow velocity $\mathbf{r}\cdot\mathbf{j}/r = \mathrm{Im}[|\mathbf{c}\cdot\mathbf{r}|^2]/r = 0$. This is the approximation in which the flow lines are circles [4]. In order to go further, it is necessary to include the quadratic and cubic terms in (7).

From (7) and (3), the right-hand side of the flow line equation (6) can be calculated up through terms of order r^3. The lowest term, of order r, vanishes; this is the circular limit just discussed. The terms of order r^2 and r^3 depend periodically on the azimuth ϕ (cf (1)), implying that the flow lines are slightly non-circular. Spiralling is determined by the secular behaviour, that is by the average variation in r over many windings. This is obtained by solving the ϕ-averaged equation, defined by

$$\left\langle \frac{\mathrm{d}r}{\mathrm{d}\phi} \right\rangle \equiv \frac{1}{2\pi} \int_0^{2\pi} \mathrm{d}\phi \frac{\mathrm{d}r}{\mathrm{d}\phi}. \tag{8}$$

A long but elementary calculation shows that the ϕ-averaged coefficient of r^2 is zero and the ϕ-averaged coefficient of r^3 is a constant, so that

$$\left\langle \frac{\mathrm{d}r}{\mathrm{d}\phi} \right\rangle = Kr^3 + \cdots. \tag{9}$$

This quantifies the slow spiralling: outwards with increasing ϕ when $K > 0$, and inwards when $K < 0$, with the radial separation between windings ($\Delta\phi = 2\pi$) decreasing as r^3 near the vortex.

Replacing r by its average (equivalent to neglecting oscillations of order r^2), and solving the resulting equation, we obtain the following formula for the flow line through the point $\{r_0, \phi_0\}$:

$$r(\phi) = \frac{r_0}{\sqrt{1 - 2Kr_0^2(\phi - \phi_0)}} + \cdots. \tag{10}$$

This is the main result of this section. It is valid when r is small, that is when $\phi \ll \phi_0 (K > 0)$ or $\phi \gg \phi_0 (K < 0)$.

The same calculation gives the explicit form of the constant K:

$$K = \frac{2\,\mathrm{Im}[(\mathrm{Tr}\,\mathbf{C})\mathbf{c}^*] \times \mathrm{Im}[\mathbf{C}^\dagger \cdot \mathbf{c}] \cdot \mathbf{e}_z}{(\mathrm{Im}[\mathbf{c}^* \times \mathbf{c} \cdot \mathbf{e}_z])^2} + \frac{3\,\mathrm{Im}\, c_i^* c_{ijj}}{2\,\mathrm{Im}[\mathbf{c}^* \times \mathbf{c} \cdot \mathbf{e}_z]}. \tag{11}$$

In general, K is not zero. However, the case mentioned earlier, that is $\nabla^2\psi(\mathbf{r}) = g(\mathbf{r})\psi(\mathbf{r})$, implies $\mathrm{Tr}\,\mathbf{C} = 0$ and $c_{ijj} = 6g(\mathbf{r})c_i$, so $K = 0$, and, as expected for this case, there is no spiralling.

In three dimensions, the phase generally varies not only around the vortex but also along it (i.e. along curl **j**—the z direction, and representing, for example, a phase factor $\exp(\mathrm{i}kz)$). Then **j** has a z component, so the flow lines migrate out of the **r** plane into a helix, reflecting the screw character of waves in the general case [3, 7, 8]. A short calculation shows that $\dot{z} \sim \mathrm{d}z/\mathrm{d}\phi \sim r^2$, so the pitch of the helix (z distance between windings) is also of order r^2. Asymptotically close to the vortex, the pitch is large compared with the inward or outward spiralling r^3 but small compared with r, so the flow lines form a tightly wound helix sweeping out a surface whose radius depends exponentially on z (because $\mathrm{d}r/r$ is proportional to $\mathrm{d}z$).

3. Flow spirals for angular momentum eigenstates

Consider the class of complex scalars of the form

$$\psi(\mathbf{r}) = \exp(\mathrm{i}m\phi)F(r). \tag{12}$$

where $F(r)$ is a complex function of the form r^m times a smooth function of r^2. These represent waves with a well-defined orbital angular momentum quantum number m [9]. The current is

$$\mathbf{j} = \mathrm{Im}[F^* F']\mathbf{e}_r + m\frac{|F|^2}{r}\mathbf{e}_\phi = \dot{r}\mathbf{e}_r + r\dot{\phi}\mathbf{e}_\phi, \tag{13}$$

and the flow line equation (6) is

$$\frac{\mathrm{d}r}{\mathrm{d}\phi} = \frac{r^2}{m}\,\mathrm{Im}\left[\frac{F'}{F}\right]. \tag{14}$$

There is no angular dependence on the right-hand side, so averaging is not necessary, and the exact flow lines can be found by integration:

$$\int_{r_0}^{r(r_0,\phi_0)} \frac{\mathrm{d}r}{r^2\,\mathrm{Im}[F'(r)/F(r)]} = \frac{(\phi-\phi_0)}{m}. \tag{15}$$

This case is both more and less general than that considered in the previous section: more general, because it allows for higher order vortices ($|m| > 1$), and (15) gives the exact flow lines not only near the vortex but also for all r; and less general, because the form (12) excludes the azimuthal modulations that were averaged away according to (8).

An example where the two cases overlap is $m = +1$ and

$$F(r) = r(1 + \mathrm{i}br^2), \tag{16}$$

that is

$$\psi(\mathbf{r}) = (x + \mathrm{i}y)(1 + \mathrm{i}b(x^2 + y^2)). \tag{17}$$

In (7) this corresponds to $\mathbf{c} = \{1,\ \mathrm{i}\}$, $\mathsf{C} = 0$, $c_{111} = 3c_{122} = 3c_{212} = 3c_{221} = \mathrm{i}b$, $c_{222} = 3c_{112} = 3c_{121} = 3c_{211} = -b$, whence the consistency of (14) and (9) can be established, with $K = 2b$ in (11). The solution of (15) for all r is (for real b)

$$\phi(r; r_0, \phi_0) = \phi_0 + \frac{(r^2 - r_0^2)}{4}\left[b + \frac{1}{b(rr_0)^2}\right]. \tag{18}$$

(If b had an imaginary part, there would be an additional term proportional to $\log(r/r_0)$.) From (18), we get the explicit Cartesian coordinates for the flow line through $\{r_0, \phi_0\}$, expressed parametrically as a function of r:

$$x(r; r_0, \phi_0) = r\cos[\phi(r; r_0, \phi_0)] \qquad y(r; r_0, \phi_0) = r\sin[\phi(r; r_0, \phi_0)]. \tag{19}$$

Figure 1(*a*) illustrates the slow spiralling out of the vortex, and figure 1(*b*) shows the phase contours and flow lines far from the vortex.

An example of physical interest is the wave propagating from an initial wavefront in the form of a helicoidal phase step of height 2π. Here $m = +1$, and [10]

$$F(r) = \sqrt{\frac{\pi}{8\mathrm{i}}}\exp\left(\frac{1}{4}\mathrm{i}r^2\right) r\left[J_0\left(\frac{1}{4}r^2\right) - \mathrm{i}J_1\left(\frac{1}{4}r^2\right)\right]. \tag{20}$$

(Here J denotes Bessel functions, and r is a scaled coordinate, defined in terms of the physical cylindrical radial coordinate R by $r = R\sqrt{(k/z)}$, where z is the propagation distance and k the wavenumber.) For this case, the integral in (15) was calculated exactly. Figure 2(*a*) shows the expected universal behaviour near the vortex (the same as in figure 1(*a*)).

Far from the vortex (figure 2(*b*)), a different phenomenon appears: the flow lines are asymptotic to concentric circles. This can be understood from the radial current associated with (20), namely

$$j_r(r) = \tfrac{1}{4}\pi r J_0\left(\tfrac{1}{4}r^2\right)J_1\left(\tfrac{1}{4}r^2\right). \tag{21}$$

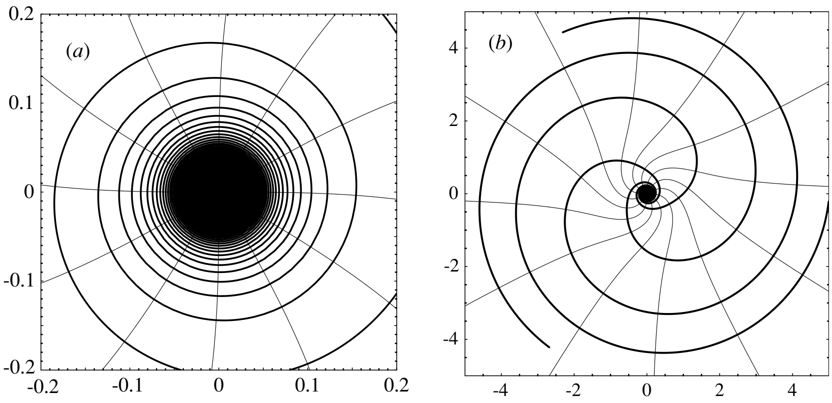

Figure 1. Thick curves: flow lines; thin curves: phase contours at intervals of $\pi/6$, for the wave (17) with $b = 1$. (*a*) Showing a single flow line spiralling out of the singularity; (*b*) as (*a*), showing three flow lines spiralling into the far field.

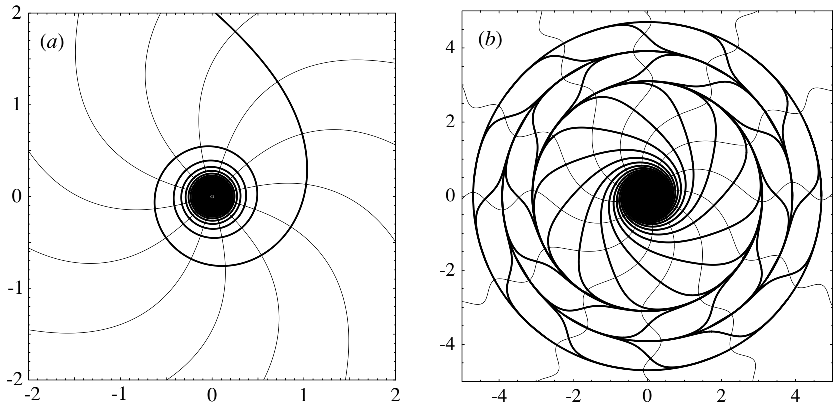

Figure 2. As figure1, for the wave (20). (*a*) A single flow line spiralling out of the singularity; (*b*) far from the singularity, showing 12 flow lines with attracting and repelling circles where the radial current vanishes.

The radial flow is alternately positive and negative (outward and inward spiralling, respectively) in zones separated by circles determined by the interlaced zeros of J_0 and J_1. Since the angular flow is always in the direction of increasing ϕ, these circles are alternately attractors and repellers.

4. Flow lines near phase extrema and phase saddles

The local form of the phase near a stationary point of phase (not to be confused with a vortex) at $\mathbf{r} = 0$ is

$$\chi(\mathbf{r}) = \chi_0 + \tfrac{1}{2}\mathbf{r}\cdot\mathbf{A}\cdot\mathbf{r} = \chi_0 + \tfrac{1}{2}A_{11}x^2 + A_{12}xy + \tfrac{1}{2}A_{22}y^2. \tag{22}$$

Positive-definite or negative-definite $\mathbf{A}$ (both eigenvalues positive or negative) correspond to phase minima or maxima respectively, while negative-definite $\mathbf{A}$ corresponds to a phase saddle [11, 12].

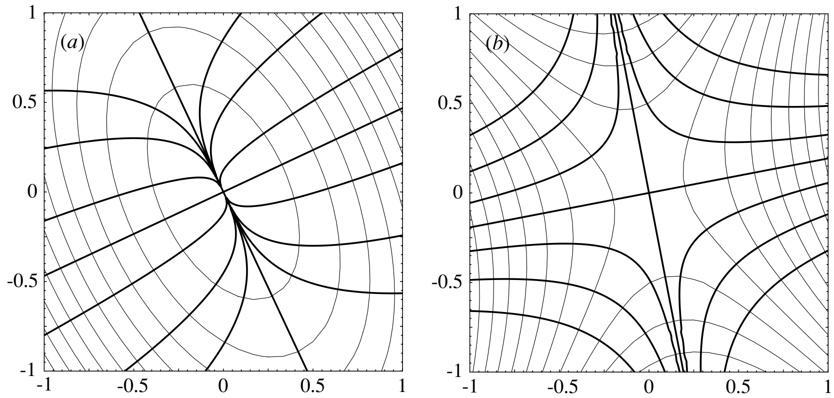

Figure 3. Curves as in figure 1, near stationary points of phase characterized by (22), with flow lines calculated using (26). (*a*) A phase minimum (current source) with $A_{11} = 1$, $A_{12} = 0.3$, $A_{22} = 0.5$; (*b*) a phase saddle with $A_{11} = 1$, $A_{12} = 0.3$, $A_{22} = -0.5$.

The flow velocity is proportional to

$$\dot{\mathbf{r}} = \nabla\chi = \mathbf{A}\cdot\mathbf{r}, \tag{23}$$

whose solution, starting from the initial point $\mathbf{r}_0$, is the matrix exponential

$$\mathbf{r}(t) = \exp(\mathbf{A}t)\mathbf{r}_0. \tag{24}$$

Expressing this in terms of the eigenvalues λ_1, λ_2 and corresponding eigenvectors u_1, u_2 of $\mathbf{A}$ leads to

$$\mathbf{u}_1\cdot\mathbf{r}(t) = \exp(\lambda_1 t)\mathbf{u}_1\cdot\mathbf{r}_0, \qquad \mathbf{u}_2\cdot\mathbf{r}(t) = \exp(\lambda_2 t)\mathbf{u}_2\cdot\mathbf{r}_0, \tag{25}$$

whence t can be eliminated, and the flow line through $\mathbf{r}_0$ written explicitly as

$$\left(\frac{\mathbf{u}_1\cdot\mathbf{r}}{\mathbf{u}_1\cdot\mathbf{r}_0}\right)^{\lambda_2} = \left(\frac{\mathbf{u}_2\cdot\mathbf{r}}{\mathbf{u}_2\cdot\mathbf{r}_0}\right)^{\lambda_1}. \tag{26}$$

In general, the exponents λ_1 and λ_2 are different, so the flow lines are non-analytic at $\mathbf{r} = 0$, unlike the phase contours, which are ellipses for extrema and hyperbolas for saddles. Figure 3 illustrates a typical case of each kind.

In the special case $A_{12} = 0$, the non-analyticity can be displayed in a simple form:

$$\left(\frac{x}{x_0}\right)^{A_{22}} = \left(\frac{y}{y_0}\right)^{A_{11}}. \tag{27}$$

Extrema correspond to $A_{11}/A_{22} > 0$, and saddles to $A_{11}/A_{22} < 0$.

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